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CES Working Papers

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Jean-Marc BONNISSEAU, Orntangar NGUENAMADJI

2008.102

Version révisée

Maison des Sciences Économiques, 106-112 boulevard de L'Hôpital, 75647 Paris Cedex 13

http://ces.univ-paris1.fr/cesdp/CES-docs.htm

ISSN : 1955-611X

On the uniqueness of local equilibria

Jean-Marc Bonnisseau and Orntangar Nguenamadji*

December 2009

Abstract

The purpose of this paper is to define a new notion of local equilibrium in an exchange economy, where the consumers face lower bounds on net trades. Then, we show that the local equilibrium is unique if the lower bounds are closed enough to 0. By the way, we also provides a convergence result of local equilibrium price toward Walras equilibrium price of a suitable linear economy.

JEL classification: C62, D50, D62.

Keywords: General economic equilibrium, local equilibria, uniqueness

1 Introduction

In an exchange economy, under suitable assumptions on the differentiability of the utility functions, it is well known that equilibrium is generically locally unique (See, for example, [5, 6, 8, 1]). This local uniqueness means that there exists only one equilibrium in a sufficiently small neighborhood around an equilibrium price.

In this paper, we focus our attention to another approach with a new concept of local equilibrium. Actually, we assume that the consumers do not consider all possible consumptions but only those which are close to the initial endowments. So the net trades on the market remain small. Formally, this means that the consumption set of a consumer is the set of consumptions which are above a given fraction of the initial endowments. We can also interpret this as the fact that the consumers face a restricted market participation where the trades must lie in bounded below subsets depending on endowments with a lower bound close to 0. Then, a local equilibrium is a Walras equilibrium of this economy with restricted consumption sets.

*Paris School of Economics, Université Paris 1 Panthéon–Sorbonne, Centre d'Economie de la Sorbonne–CNRS, 106-112 Boulevard de l'Hôpital, 75647 Paris Cedex 13, France. E-mail: `Jean-Marc.Bonnisseau@univ-paris1.fr`, `Orntangar.Nguenamadji@univ-paris1.fr`

We first study the link between local equilibria and equilibria of a family of linear economies as the tangent linear economies introduced in [4, 3]. The linear economies are defined by the constant marginal utilities of the agents, which are computed using the marginal utilities at the initial endowments.

Since the trades in a local economy converge to 0 when the trades are more and more restricted, we have to rescale them in order to be comparable with the ones of the linear economies. Actually, we show that we can associate to a local economy an auxiliary economy with the same initial endowments, the positive orthant for the consumption sets and new utility functions derived from the original one by a rescaling of the net trade. Then, we show that the equilibria of these auxiliary economies converge to an equilibrium of a linear economy when the scaling factors tend to 0. The limit linear economy depends on the limit of the direction of the lower bounds on the net trades. In particular, this implies that a local equilibrium price is close to the unique equilibrium price of the limit linear economy and the rescaled equilibrium allocations are also close to the equilibrium allocations of the tangent linear economy.

The main contribution of the paper is to show that the local equilibrium price is unique for sufficiently restricted trade when the utility functions are strictly quasi-concave. Obviously, we also have the uniqueness of the equilibrium allocation. The proof is based on a concave representation of preferences and a convexity-like property of the indirect utility functions.

This result can be related to the fact that the Walras equilibrium price is unique when the initial endowments are in a neighborhood of the contract curve or of the set of Pareto optimal allocations (See [1]). In that case, without any a priori restrictions, the trades are small. So we remain in the same kind of interpretation: the equilibrium is unique when the trades are small, that is when the gains in terms of utility level are closed to the first order linear approximation.

This result also provides a new result about uniqueness of Walras equilibria when the second derivatives of utility functions are close to 0. This generalizes the result of uniqueness for linear exchange economy, where the second derivatives of utility functions are equal to 0.

Consequently, we have two approaches of the local uniqueness of equilibria. Either, as in the literature, we consider a neighborhood of the price and we require that the equilibrium price is unique in this neighborhood or we consider restricted trades, that is equilibrium allocations in a neighborhood of the initial endowments and we have a unique equilibrium price.

This paper is a first step toward the definition of a discrete Walrasian exchange process where the exchanges are determined at each period by the unique net trades associated to the unique local equilibrium.

2 Local equilibrium

We consider a family of pure exchange economies with ℓ commodities and m consumers[1]. The preferences of the agents are represented by utility functions u_i from $\mathbb{R}^\ell_+$ to $\mathbb{R}$. The initial endowments are taken in a nonempty compact subset $\mathbf{E}$ of $(\mathbb{R}^\ell_{++})^m$. So, for an element $\mathbf{e} = (e_i)_{i=1}^m \in \mathbf{E}$, the economy $\mathcal{E}(\mathbf{e})$ is defined by $(\mathbb{R}^\ell_+, u_i, e_i)_{i=1}^m$. For $\tau = (\tau_i)_{i=1}^m \in ([0;1]^\ell)^m$, we define a τ-local equilibrium as follows:

Definition 2.1 $(p^*, (x_i^*)) \in \mathbb{R}^\ell_+ \times (\mathbb{R}^\ell_+)^m$ is a τ-local equilibrium of the economy $\mathcal{E}(\mathbf{e})$ if

(i) for all $i = 1, \ldots, m$, x_i^* is a solution of $\begin{cases} \text{maximize } u_i(x_i) \\ p^* \cdot x_i \leq p^* \cdot e_i \\ x_i \geq (\mathbf{1} - \tau_i)\square e_i \end{cases}$

(ii) $\sum_{i=1}^m x_i^* = \sum_{i=1}^m e_i$.

The difference with a standard Walras equilibrium comes from the constraint $x_i \geq (\mathbf{1} - \tau_i)\square e_i$ instead of $x_i \geq 0$. This constraint means that the consumption set of the consumer is $\{(\mathbf{1} - \tau_i)\square e_i\} + \mathbb{R}^\ell_+$ instead of $\mathbb{R}^\ell_+$, so a τ-local equilibrium is a Walras equilibrium of the τ-local economy $\mathcal{E}^\tau(\mathbf{e}) = (\{(\mathbf{1} - \tau_i)\square e_i\} + \mathbb{R}^\ell_+, u_i, e_i)_{i=1}^m$. One remarks that a τ-local equilibrium is merely a Walras equilibrium of the economy $\mathcal{E}(\mathbf{e})$ when $\tau_i = \mathbf{1}$ for all i. A τ- local equilibrium is $(p^*, (e_i))$, where p^* is any non-zero price vector in $\mathbb{R}^\ell_+$ when $\tau_i = 0$ for all i.

This constraint $x_i \geq (\mathbf{1} - \tau_i)\square e_i$ is equivalent to the constraint on the net trade $z_i = x_i - e_i \geq -\tau_i \square e_i$. So a local equilibrium is an equilibrium with restricted trades, where the maximization problem of a consumer is:

$$\begin{cases} \text{maximize } u_i(e_i + z_i) \\ p^* \cdot z_i \leq 0 \\ z_i \geq -\tau_i \square e_i \end{cases}$$

When for all i and for all h $\tau_{\mathbf{i}}^{\mathbf{h}}$ is small , the equilibrium allocation of a τ-local equilibrium is close to $\mathbf{e}$. Indeed, if we combine the lower constraint $x_i \geq (\mathbf{1} - \tau_i)\square e_i$ with the market clearing condition (ii), one gets $(\mathbf{1} - \tau_i)\square e_i \leq x_i^* \leq e_i + \sum_{j \neq i} \tau_j \square e_j$, so $\tau_i \square e_i \leq x_i^* - e_i \leq \sum_{j \neq i} \tau_j \square e_j$.

We now introduce a family of auxiliary economies $(\hat{\mathcal{E}}^\tau(\mathbf{e}))_{(\tau, \mathbf{e}) \in (]0,1[^\ell)^m \times \mathbf{E}}$ and we show the link between a τ-local equilibrium of $\mathcal{E}(\mathbf{e})$ and a Walras equilibrium of the auxiliary economy $\hat{\mathcal{E}}^\tau(\mathbf{e})$. The difference between $\mathcal{E}^\tau(\mathbf{e})$ and $\hat{\mathcal{E}}^\tau(\mathbf{e})$ is that we modify the preferences instead of the consumption sets.

[1] In $\mathbb{R}^\ell$, S denotes the simplex and $\|x\| = \sum_{h=1}^\ell |x_h|$ for all $x \in \mathbb{R}^\ell$. The box product is defined as follows: for a pair of vectors (x, y) of $\mathbb{R}^\ell$, $x \square y$ is the vector of $\mathbb{R}^\ell$ with components $x^h y^h$ for $h = 1, \ldots, \ell$. We denote by $\mathbf{1}$ the vector of $\mathbb{R}^\ell$ the components of which are all equal to 1.

Definition 2.2 For all $\tau \in (]0,1[^\ell)^m$ and $\mathbf{e} \in \mathbf{E}$, the pure exchange economy $\hat{\mathcal{E}}^\tau(\mathbf{e})$ is the economy where the consumption sets are equal to $\mathbb{R}^\ell_+$, the initial endowments are $\mathbf{e} = (e_i)_{i=1}^m$ and the preferences are represented by the utility function $u_{ie_i}^{\tau_i}$ defined by:

$$u_{ie_i}^{\tau_i}(x_i) = \frac{u_i\left((1-\tau_i)\square e_i + \tau_i \square x_i)\right) - u_i(e_i)}{\|\tau_i\|}$$

Note that the coefficient $1/\|\tau_i\|$ and the additional term $-u_i(e_i)$ are useless in the definition since the preferences are the same with or without them. Nevertheless we use this formulation to emphasize later the convergence towards the tangent linear economy.

We can interpret the preferences of the agent as follows. For two consumptions x_i and x'_i, far from e_i, the consumer compares them by considering the two allocations $e_i + \tau_i \square (x_i - e_i)$ and $e_i + \tau_i \square (x'_i - e_i)$ which are close to the current allocation e_i. So, it is as if the consumer has well known preferences around her initial endowments and extrapolates them for the farther allocations. This can be understood as a limited rationality of the agent or a risk aversion for large trades. Once again, if $\tau_i = \mathbf{1}$ for all i, then $\hat{\mathcal{E}}^\tau(\mathbf{e})$ is merely the initial economy $\mathcal{E}(\mathbf{e})$.

Remark 2.1 We can remark that the definition of $\hat{\mathcal{E}}^\tau(\mathbf{e})$ is independent of the choice of the representations of preferences. That is, if $\tilde{u}_i$ represents the same preferences as u_i then $\tilde{u}_{ie_i}^{\tau_i}$ represents the same preferences as $u_{ie_i}^{\tau_i}$ since we have:

$$\begin{aligned} u_i^{\tau_i}(x_i) \geq u_i^{\tau_i}(x'_i) &\Leftrightarrow u_i(e_i + \tau_i \square (x_i - e_i)) \geq u_i(e_i + \tau_i \square (x' - e_i)) \\ &\Leftrightarrow \tilde{u}_i(e_i + \tau_i \square (x_i - e_i)) \geq \tilde{u}_i(e_i + \tau_i \square (x' - e_i)) \\ &\Leftrightarrow \tilde{u}_{ie_i}^{\tau_i}(x_i) \geq \tilde{u}_{ie_i}^{\tau_i}(x'_i). \end{aligned}$$

So, the economies with the utility functions $u_{ie_i}^{\tau_i}$ or with the utility functions $\tilde{u}_{ie_i}^{\tau_i}$ are identical.

The next proposition gives the link between τ-local equilibrium of $\mathcal{E}(\mathbf{e})$ and Walras equilibrium of $\hat{\mathcal{E}}^\tau(\mathbf{e})$ for each $\tau \in (]0,1[^\ell)^m$ and $\mathbf{e} \in \mathbf{E}$.

Proposition 2.1 *Let $\mathbf{e} \in \mathbf{E}$ and $\tau \in (]0,1[^\ell)^m$. If $(p^*, (x_i^*))$ is a τ-local equilibrium of $\mathcal{E}(\mathbf{e})$, then $\left(p^*, \left((\tau_i)^{-1} \square (x_i^* - e_i)\right)_i\right)$ is an equilibrium of the economy $\hat{\mathcal{E}}^\tau(\mathbf{e})$. Conversely, if $(p^*, (\xi_i^*))$ is an equilibrium of $\hat{\mathcal{E}}^\tau(\mathbf{e})$, then $(p^*, (e_i + \tau_i \square (\xi_i^* - e_i)_i)$ is a τ-local equilibrium of the economy $\mathcal{E}(\mathbf{e})$.*

The proof is obvious starting from the definitions of a Walras equilibrium and τ-local equilibrium.

We now end this section by considering the tangent linear economy $\hat{\mathcal{E}}_0^\rho(\mathbf{e})$ associated to $\mathcal{E}(\mathbf{e})$ and to a parameter $\rho \in S^m$. This follows the approach of [4].

Definition 2.3 If the utility functions are differentiable at the initial endowments $\mathbf{e} = (e_i)$, the linear economy $\hat{\mathcal{E}}_0^{\rho}(\mathbf{e})$ is the economy with the consumption sets $\mathbb{R}_+^{\ell}$, the initial endowments e_i and the linear preferences represented by the utility function $\hat{u}_{ie_i}^{\rho_i}$ defined by:

$$\hat{u}_{ie_i}^{\rho_i}(x_i) = [\rho_i \square \nabla u_i(e_i)] \cdot (x_i - e_i)$$

Once again, we remark that the preferences are represented by the simpler utility function $[\rho_i \square \nabla u_i(e_i)] \cdot x_i$ but we choose such formulation to get simpler statements later.

3 Convergence of τ-local equilibria

From now on, we consider $\tau \in (]0, \frac{1}{2}]^{\ell})^m$. The upper bound $\frac{1}{2}$ is arbitrary chosen in $]0, 1[$ to guarantee the compactness of the sets we are considering in the proofs. Recall that $\mathbf{E}$ is a compact of $(\mathbb{R}_{++}^{\ell})^m$.

We posit the following standard assumption on the preferences in the theory of general economic equilibrium from a differentiable approach.

Assumption C. For all i, u_i is twice continuously differentiable on $\mathbb{R}_{++}^{\ell}$ and $\nabla u_i(x_i) \in \mathbb{R}_{++}^{\ell}$ for all $x_i \in \mathbb{R}_{++}^{\ell}$.

The following propositions show that the distance between the economies $\hat{\mathcal{E}}^{\tau}(\mathbf{e})$ and $\hat{\mathcal{E}}_0^{\rho}(\mathbf{e})$ converges to zero when τ converges uniformly to zero.

Proposition 3.1 *Let $\mathbf{K}$ be a compact subset of $(\mathbb{R}_+^{\ell})^m$. Under Assumption C, for all $\varepsilon > 0$, there exists $\bar{\theta} > 0$ such that for all $\tau \in ([0, 1/2]^{\ell})^m$, for all $(\mathbf{e}, x) \in \mathbf{E} \times \mathbf{K}$ and for all i, if $0 < \|\tau_i\| \leq \bar{\theta}$, then*

$$|u_{ie_i}^{\tau_i}(x_i) - \hat{u}_{ie_i}^{\rho_i}(x_i)| \leq \varepsilon$$

where $\rho_i = \frac{1}{\|\tau_i\|}\tau_i$.

The proof of this proposition given in Appendix is a direct consequence of Taylor formula. We deduce from this uniform convergence on compacta a result of continuity (see the proof in Appendix).

Corollary 3.1 *Under Assumption C, for all i, the function U from*

$$U: \ [0, \frac{1}{2}]^m \times S^m \times \mathbf{E} \times (\mathbb{R}_+^{\ell})^m \to \mathbb{R}^m$$

defined by

$$U_i(\theta, \rho, \mathbf{e}, x) = \begin{cases} u_{ie_i}^{\theta_i \rho_i}(x_i) & \text{if } \theta_i \neq 0 \\ \hat{u}_{ie_i}^{\rho_i}(x_i) & \text{if } \theta_i = 0 \end{cases}$$

is continuous.

We now deduce the convergence of equilibrium prices and equilibrium allocations.

In the following, we normalize the prices in the simplex S of $\mathbb{R}^\ell$. For $\tau \in \left(]0, \frac{1}{2}]^\ell\right)^m$, we denote by P^τ the normalized price equilibrium correspondence associated to $\hat{\mathcal{E}}^\tau$. For $\rho \in S^m$, we denote by P_0^ρ the normalized price equilibrium correspondence associated to the linear economy $\hat{\mathcal{E}}_0^\rho$. We know from [4, 7] that $(\rho, \mathbf{e}) \to P_0^\rho(\mathbf{e})$ is actually a single valued continuous mapping on $(S \cap \mathbb{R}^\ell_{++})^m \times (\mathbb{R}^\ell_{++})^m$. Let $\Gamma \subset \mathbb{R}^\ell_{++} \cup \{0\}$ be a nonempty closed convex cone and $\Sigma = \Gamma \cap S$.

Proposition 3.2 *Under Assumption C, for all $\varepsilon > 0$ there exists $\hat{\theta} > 0$ such that for all $\tau \in (\Gamma \cap]0, \hat{\theta}]^\ell)^m$ and for all $\mathbf{e} \in \mathbf{E}$,*

$$P^\tau(\mathbf{e}) \subset \bar{B}(P_0^\rho(\mathbf{e}), \varepsilon)$$

where $\bar{B}(P_0^\rho(\mathbf{e}), \varepsilon)$ denotes the closed ball of center $P_0^\rho(\mathbf{e})$ and radius ε and $\rho = \left(\frac{1}{\|\tau_i\|}\tau_i\right)_{i=1}^m$.

Remark 3.1 Note that the above proposition together with the continuity of the mapping P_0^ρ implies that for all sequences $(\tau^\nu, \mathbf{e}^\nu, p^\nu) \in (\Gamma \cap]0, 1/2]^\ell)^m \times \mathbf{E} \times S$ converging to $(0, \mathbf{e}, p)$ and satisfying $p^\nu \in P^{\tau^\nu}(\mathbf{e}^\nu)$ for all integer ν, if $\rho^\nu = \left(\frac{1}{\|\tau_i^\nu\|}\tau_i^\nu\right)_{i=1}^m$ converges to ρ, then $p \in P_0^\rho(\mathbf{e})$.

Proof. Assume on the contrary that there exists $\varepsilon > 0$ such that for all integer $\nu \geq 1$, there exists $\tau^\nu \in (\Gamma \cap]0, 1/\nu]^\ell)^m$ and $\mathbf{e}^\nu \in \mathbf{E}$ such that $P^{\tau^\nu}(\mathbf{e}^\nu) \not\subseteq \bar{B}(P_0^{\rho^\nu}(\mathbf{e}^\nu), \varepsilon)$ where $\rho^\nu = \left(\frac{1}{\|\tau_i^\nu\|}\tau_i^\nu\right)_{i=1}^m$. Note that ρ^ν belongs to Σ^m. Then, for all integer ν, there exists $\bar{x}^\nu \in (\mathbb{R}^\ell_+)^m$ and $\bar{p}^\nu \in P^{\tau^\nu}(\mathbf{e}^\nu)$ such that $(p^\nu, \bar{x}^\nu)$ is an equilibrium of $\hat{\mathcal{E}}^{\tau^\nu}(\mathbf{e}^\nu)$ and $p^\nu \notin \bar{B}(P_0^{\rho^\nu}(\mathbf{e}^\nu), \varepsilon)$.

The sequence $(\bar{x}^\nu)$ is bounded since $\mathbf{E}$ is compact and $\bar{x}_i^\nu \leq \sum_{i=1}^m e_i^\nu$ for all ν. Since Σ, $\mathbf{E}$ and S are compact subsets, there exists a subsequence of $(\tau^\nu, \rho^\nu, \mathbf{e}^\nu, \bar{p}^\nu, \bar{x}^\nu)$ (denoted identically for the sake of simpler notations) which converges. Let us denote $(0, \rho, \mathbf{e}, \bar{p}, \bar{x})$ its limit. From the fact that $(\bar{p}^\nu, \bar{x}^\nu)$ is an equilibrium of $\hat{\mathcal{E}}^{\tau^\nu}(\mathbf{e}^\nu)$, one deduces that $\sum_{i=1}^m \bar{x}_i = \sum_{i=1}^m e_i$ and $\bar{p} \cdot \bar{x}_i \leq \bar{p} \cdot e_i$ for all i. Furthermore, $\bar{p} \notin B(P_0^\rho(\mathbf{e}), \varepsilon)$. Hence $\bar{p}$ is not an equilibrium price vector of $\hat{\mathcal{E}}_0^\rho(\mathbf{e})$, so there exists i such that $\bar{x}_i$ does not belong to the demand for $\hat{u}_{ie_i}^{\rho_i}$ at $(\bar{p}, \bar{p} \cdot e_i)$.

Consequently there exists a vector $\xi_i \in \mathbb{R}^\ell_+$ such that $\bar{p} \cdot \xi_i \leq \bar{p} \cdot e_i$ and $\hat{u}_{ie_i}^{\rho_i}(\xi_i) > \hat{u}_{ie_i}^{\rho_i}(\bar{x}_i)$. Since according to Assumption C, $\nabla u_i(e_i) \gg 0$ and $\bar{p} \cdot e_i > 0$, we can choose ξ_i in such a way that $\bar{p} \cdot \xi_i < \bar{p} \cdot e_i$.

For ν large enough, $p^\nu \cdot \xi_i < p^\nu \cdot e_i^\nu$. Since $(\bar{p}^\nu, \bar{x}^\nu)$ is an equilibrium of $\hat{\mathcal{E}}^{\tau^\nu}(\mathbf{e}^\nu)$, one deduces that $u_{ie_i^\nu}^{\tau_i^\nu}(\xi_i) < u_{ie_i^\nu}^{\tau_i^\nu}(\bar{x}_i^\nu)$. From Corollary 3.1, since $\tau_i^\nu = \|\tau_i^\nu\|\rho_i^\nu$, one deduces that $\lim_{\nu \to \infty} u_{ie_i}^{\|\tau_i^\nu\|\rho_i^\nu}(\xi_i) = \hat{u}_{ie_i}^{\rho_i}(\xi_i) \leq \lim_{\nu \to \infty} u_{ie_i}^{\|\tau_i^\nu\|\rho_i^\nu}(\bar{x}_i^\nu) = \hat{u}_{ie_i}^{\rho_i}(\bar{x}_i)$, which is in contradiction with $\hat{u}_{ie_i}^{\rho_i}(\xi_i) > \hat{u}_{ie_i}^{\rho_i}(\bar{x}_i)$. $\square$

We now state a result on the convergence of equilibrium allocations. Since there is no uniqueness of the equilibrium allocation for the linear tangent economy, we do not really have a convergence property but a closed graph property.

Proposition 3.3 *Let $(\tau^\nu, \mathbf{e}^\nu, x^\nu)$ be a sequence of $(\Gamma \cap]0, \frac{1}{2}]^\ell)^m \times \mathbf{E} \times (\mathbb{R}^\ell_+)^m$ converging to $(0, \mathbf{e}, \bar{x})$. Under Assumption C, if x^ν is an equilibrium allocation of the economy $\hat{\mathcal{E}}^{\tau^\nu}(\mathbf{e}^\nu)$ for all ν and $\rho^\nu = \left(\frac{1}{\|\tau_i^\nu\|}\tau_i^\nu\right)_{i=1}^m$ converges to ρ, then $\bar{x}$ is an equilibrium allocation of the tangent linear economy $\hat{\mathcal{E}}^\rho(\mathbf{e})$.*

Proof. Let $(\tau^\nu, \mathbf{e}^\nu, x^\nu)$ be a sequence of $(\Gamma \cap]0, \frac{1}{2}]^\ell)^m \times \mathbf{E} \times (\mathbb{R}^\ell_+)^m$ converging to $(0, \mathbf{e}, \bar{x})$. For all ν, let us assume that x^ν is an equilibrium allocation of the economy $\hat{\mathcal{E}}^{\tau^\nu}(\mathbf{e}^\nu)$. Let us assume that $\rho^\nu = \left(\frac{1}{\|\tau_i^\nu\|}\tau_i^\nu\right)_{i=1}^m$ converges to ρ.

Let (p^ν) be the sequence of normalized equilibrium price associated to (x^ν) and $\bar{p} = P_0^\rho(\mathbf{e})$. From Proposition 3.2, (p^ν) converges to $\bar{p}$. Consequently, one easily checks that $\bar{p} \cdot \bar{x}_i = \bar{p} \cdot e_i$ and $\sum_{i=1}^m \bar{x}_i = \sum_{i=1}^m e_i$.

Since $e_i \gg 0$ and $\bar{p} \in S$, $\bar{p} \cdot e_i > 0$. Let $\xi_i \in \mathbb{R}^\ell_+$ such that $\bar{p} \cdot \xi_i < \bar{p} \cdot e_i$. For ν large enough, $p^\nu \cdot \xi_i \leq p^\nu \cdot e_i^\nu$. Hence, since (p^ν, x^ν) is an equilibrium of $\hat{\mathcal{E}}^{\tau^\nu}(\mathbf{e}^\nu)$, one has $u_{ie_i^\nu}^{\tau_i^\nu}(\xi_i) \leq u_{ie_i^\nu}^{\tau_i^\nu}(x_i^\nu)$. From Corollary 3.1, one gets at the limit $\hat{u}_{ie_i}^\rho(\xi_i) \leq \hat{u}_{ie_i}^\rho(\bar{x}_i)$. Now, let $\xi_i \in \mathbb{R}^\ell_+$ such that $\bar{p} \cdot \xi_i \leq \bar{p} \cdot e_i$. Since $\bar{p} \cdot e_i > 0$, ξ_i is the limit of a sequence of consumptions ξ_i^μ such that $\bar{p} \cdot \xi_i^\mu < \bar{p} \cdot e_i$. From the continuity of the utility function $\hat{u}_{ie_i}^\rho$, one concludes that $\hat{u}_{ie_i}^\rho(\xi_i) \leq \hat{u}_{ie_i}^\rho(\bar{x}_i)$. So $\bar{x}_i$ is optimal in the budget set associated to $\bar{p}$ and $\bar{p} \cdot e_i$. Hence $(\bar{x}_i)$ is an equilibrium allocation of the economy $\hat{\mathcal{E}}_0^\rho(\mathbf{e})$. □

4 Uniqueness of the local equilibrium

The main result of the paper is the following one about the uniform uniqueness of the τ-local equilibrium for initial endowments remaining in a compact subset of $(\mathbb{R}^\ell_{++})^m$ if the vector τ is uniformly small enough. For this, we need the utility functions to satisfy, in addition to C, the following standard assumption.

Assumption C'**.** For all i, for all $x_i \in \mathbb{R}^\ell_{++}$ and for all $z_i \in \mathbb{R}^\ell \setminus \{0\}$, one has :

$[\nabla u_i(x_i) \cdot z_i = 0] \Rightarrow [z_i \cdot D^2 u_i(x_i)(z_i) < 0]$.

Proposition 4.1 *Under Assumptions C and C', there exists $\tilde{\theta} > 0$ such that, for all $\tau \in (\Gamma \cap]0, \tilde{\theta}]^\ell)^m$ and for all $\mathbf{e} \in \mathbf{E}$ the economy $\hat{\mathcal{E}}^\tau(\mathbf{e})$ has a unique normalized equilibrium. Hence, for all $\tau \in (\Gamma \cap]0, \tilde{\theta}]^\ell)^m$ and for all $\mathbf{e} \in \mathbf{E}$, the economy $\mathcal{E}(\mathbf{e})$ has a unique normalized τ-local equilibrium.*

The proof is based on a combination of the concavity of the utility function and a convexity-like property of the indirect utility functions associated to

$u^{\tau}_{ie_i}$ when the vector τ is small enough. This property is known to be satisfied when the utility function is linear (See, [2]).

In order to prepare this proof, we first recall that we can consider concave utility functions without any loss of generality. Let

$$\mathcal{A}(\mathbf{E}) = \{(x_i) \in (\mathbb{R}^{\ell}_{+})^m \mid \exists \mathbf{e} \in \mathbf{E}, \sum_{i=1}^{m} x_i = \sum_{i=1}^{m} e_i\}$$

Since $\mathbf{E}$ is compact, $\mathcal{A}(\mathbf{E})$ is a compact subset of $(\mathbb{R}^{\ell}_{+})^m$. Let K be the image of $([0,(1/2)]^{\ell})^m \times \mathbf{E} \times \mathcal{A}(\mathbf{E})$ by the continuous mapping $(\tau, \mathbf{e}, (x_i))$ to $((1-\tau_i)\square e_i + \tau_i \square x_i)_{i=1}^m$. Since $([0,(1/2)]^{\ell})^m \times \mathbf{E} \times \mathcal{A}(\mathbf{E})$ is compact, K is compact. We remark that $K \subset (\mathbb{R}^{\ell}_{++})^m$. Finally, for all i, $\hat{K}_i$ is the projection of K on the i-th component of $(\mathbb{R}^{\ell})^m$, that is

$$\hat{K}_i = \{x_i \in \mathbb{R}^{\ell}_{++} \mid \exists x_{-i} \in (\mathbb{R}^{\ell}_{++})^{m-1}, (x_i, x_{-i}) \in K\}$$

We now apply the following lemma (See, [8] Proposition 2.6.4 p.80).

Lemma 4.1 *Under Assumptions C and C′, there exists a C^2 utility function $\tilde{u}_i$ on $\mathbb{R}^{\ell}_{++}$, which represents the same preferences as u_i and which is strictly concave on a convex open neighborhood of $\hat{K}_i$.*

Remark. Thanks to Lemma 4.1, without any loss of generality, we assume that the utility function u_i are concave on a convex open neighborhood of $\hat{K}_i$ in the remainder of the paper.

We are now working on the indirect utility function. For all $\tau_i \in (]0, \frac{1}{2}])^{\ell}$, the indirect utility function $v^{\tau_i}_{ie_i}$ associated to $u^{\tau_i}_{ie_i}$ is defined by:

$$v^{\tau_i}_{ie_i}(p, w) = \max\{u^{\tau_i}_{ie_i}(x_i) \mid p \cdot x_i \leq w, x_i \geq 0\}$$

Here, we introduce two functions.The first one is denoted by ψ and defined from $\mathbb{R}^{\ell}$ to $\mathbb{R}^{\ell}_{++}$ by:

$$\psi(q) = (\exp(q_1), \exp(q_2), \ldots, \exp(q_{\ell})).$$

The second one is denoted by $V^{\tau_i}_{ie_i}$ and defined from $\mathbb{R}^{\ell}$ to $\mathbb{R}$ by:

$$V^{\tau_i}_{ie_i}(q) = v^{\tau_i}_{ie_i}(\psi(q), \psi(q) \cdot e_i).$$

Before stating the key lemma on the convexity of $V^{\tau_i}_{ie_i}$, we first define a compact subset of potential equilibrium prices. As already noticed, the function $(\rho, \mathbf{e}) \to P^{\rho}_0(\mathbf{e})$ is a continuous mapping on $(S \cap \mathbb{R}^{\ell}_{++})^m \times \mathbf{E}$. So the image $\mathbb{P}_0$ of $\Sigma^m \times \mathbf{E}$ by P_0 is a compact subset of $\mathbb{R}^{\ell}_{++}$. Let $\varepsilon > 0$ small enough, such that $\bar{B}(\mathbb{P}_0, \varepsilon) \subset \mathbb{R}^{\ell}_{++}$. Then, from Proposition 3.2, there exists $\hat{\theta} > 0$ such that for all $(\tau, \mathbf{e}) \in (\Gamma \cap]0, \hat{\theta}]^{\ell})^m \times \mathbf{E}$, $P^{\tau}(\mathbf{e}) \subset \bar{B}(\mathbb{P}_0, \varepsilon)$. Let

$$Q = \{(\ln(p_1/p_{\ell}), \ldots, \ln(p_{\ell-1}/p_{\ell}), 0) \mid p \in \bar{B}(\mathbb{P}_0, \varepsilon)\}$$

Q is a compact subset of $\mathbb{R}^{\ell}$. The proof of the following key lemma is given in Appendix.

Lemma 4.2 *Under Assumption C and C′, there exists $\overline{\theta} \in]0, \frac{1}{2}]$ such that for all $(\tau, \mathbf{e}) \in (\Gamma \cap]0, \overline{\theta}]^\ell)^m \times \mathbf{E}$, for all i, the function $V_{ie_i}^{\tau_i}$ is strictly convex on the convex hull of Q.*

Proof of Proposition 4.1

Let $\tilde{\theta} = \min\{\hat{\theta}, \overline{\theta}\}$ where $\overline{\theta}$ is given by Lemma 4.2 and $\hat{\theta}$ by Proposition 3.2. Let us assume by contraposition that there exists $(\tau, \mathbf{e}) \in (\Gamma \cap]0, \tilde{\theta}]^\ell)^m \times \mathbf{E}$ such that the economy $\hat{\mathcal{E}}^\tau(\mathbf{e})$ has two different normalized equilibrium prices p and p'. Let (x_i) (resp. (y_i)) be the equilibrium allocation associated to p (resp. p'). Let r be the price vector defined by $r_h = \sqrt{p_h p'_h}$ for $h = 1, \ldots, \ell$. Let q and q' be the vectors defined as follows:

$$q = (\ln(p_1/p_\ell), \ldots, \ln(p_{\ell-1}/p_\ell), 0)$$

$$q' = \left(\ln(p'_1/p'_\ell), \ldots, \ln(p'_{\ell-1}/p'_\ell), 0\right)$$

Since $\mathbf{e} \in \mathbf{E}$, $\tau \in (\Gamma \cap]0, \tilde{\theta}]^\ell)^m \subset (\Gamma \cap]0, \hat{\theta}]^\ell)^m$, q and q' belongs to Q. So we deduce from Lemma 4.2 that for all i,

$$V_{ie_i}^{\tau_i}\left(\frac{1}{2}q + \frac{1}{2}q'\right) < \frac{1}{2}V_{ie_i}^{\tau_i}(q) + \frac{1}{2}V_{ie_i}^{\tau_i}(q').$$

In addition, one remarks that for all h, $\ln(r_h/r_\ell) = \frac{1}{2}q_h + \frac{1}{2}q'_h$, $V_{ie_i}^{\tau_i}(q) = v_{ie_i}^{\tau_i}(p)$ and $V_{ie_i}^{\tau_i}(q') = v_{ie_i}^{\tau_i}(p')$. Finally we get $\dfrac{1}{2}v_{ie_i}^{\tau_i}(p) + \dfrac{1}{2}v_{ie_i}^{\tau_i}(p') > v_{ie_i}^{\tau_i}(r)$ for all i. So, since (x_i) (resp. (y_i)) is the equilibrium allocation associated to p (resp. p'), we get, for all i,

$$\frac{1}{2}\tilde{u}_{ie_i}^{\tau}(x_i) + \frac{1}{2}\tilde{u}_{ie_i}^{\tau}(y_i) > v_{ie_i}^{\tau}(r).$$

From the definition of $\mathcal{A}(\mathbf{E})$ and K, $e_i + \tau_i \square (x_i - e_i)$ and $e_i + \tau_i \square (y_i - e_i)$ belong to $\hat{K}_i$. Since $\tilde{u}_{ie_i}^{\tau_i}(\xi_i) = (1/\|\tau\|)\left(\tilde{u}_i(e_i + \tau_i \square (\xi_i - e_i)) - \tilde{u}_i(e_i)\right)$ and $(1/2)(e_i + \tau_i \square (x_i - e_i)) + (1/2)(e_i + \tau_i \square (y_i - e_i)) = e_i + \tau_i \square (((1/2)x_i + (1/2)y_i) - e_i)$, the concavity of u_i implies that:

$$\tilde{u}_{ie_i}^{\tau_i}\left(\frac{1}{2}x_i + \frac{1}{2}y_i\right) \geq \frac{1}{2}\tilde{u}_{ie_i}^{\tau_i}(x_i) + \frac{1}{2}\tilde{u}_{ie_i}^{\tau_i}(y_i) > v_i^{\tau_i}(r) \quad \forall i.$$

From the definition of the indirect utility function, this implies $r \cdot (\frac{1}{2}x_i + \frac{1}{2}y_i) > r \cdot e_i$ for all i. We get a contradiction since

$$\sum_{i=1}^m x_i = \sum_{i=1}^m y_i = \sum_{i=1}^m e_i.$$

The utility function u_i being strictly quasi-concave on $\mathbb{R}^\ell_{++}$ by Assumption C', the utility function $u_{ie_i}^{\tau_i}$ is strictly quasi-concave on $\mathbb{R}^\ell_+$ for all $\tau_i \in]0; 1[^\ell$. So the demand is single valued and the uniqueness of the normalized equilibrium price implies the uniqueness of the equilibrium allocation. □

For all $(\mathbf{e}, \tau) \in \mathbf{E} \times (\Gamma \cap]0, (1/2)]^{\ell})^m$, we denote by $X(\mathbf{e}, \tau)$ the set of normalized τ-local equilibrium of the economy $\mathcal{E}(\mathbf{e})$. The previous result shows that X is a mapping on $\mathbf{E} \times (\Gamma \cap]0, \tilde{\theta}]^{\ell})^m$.

Corollary 4.1 *Under assumption C and C′, the normalized τ-local equilibrium mapping X is continuous on $\mathbf{E} \times (\Gamma \cap]0, \tilde{\theta}]^{\ell})^m$.*

Proof. Since the range of X is included in $S \times \mathcal{A}(\mathbf{E})$, which is bounded, it suffices to show that for all sequence $(\mathbf{e}^\nu, \tau^\nu)$ in $\mathbf{E} \times (\Gamma \cap]0, \tilde{\theta}]^{\ell})^m$ converging to $(\mathbf{e}, \tau) \in \mathbf{E} \times (\Gamma \cap]0, \tilde{\theta}]^{\ell})^m$ such that the sequence $(X(\mathbf{e}^\nu, \tau^\nu))$ converges to $(\overline{p}, (\overline{x}_i))$, then $X(\mathbf{e}, \tau) = (\overline{p}, (\overline{x}_i))$.

For all ν, let $X(\mathbf{e}^\nu, \tau^\nu) = (p^\nu, (x_i^\nu))$. Since $x_i^\nu \geq (1-\tau^\nu)\square e_i^\nu$, $p^\nu \cdot x_i{}^\nu \leq p^\nu \cdot e_i^\nu$ for all i, and $\sum_{i=1}^m x_i^\nu = \sum_{i=1}^m e_i^\nu$ for all ν, we get $\overline{x}_i \geq (1-\tau)\square e_i$, $\overline{p} \cdot \overline{x}_i \leq \overline{p} \cdot e_i$ for all i, and $\sum_{i=1}^m \overline{x_i} = \sum_{i=1}^m e_i$.

Since $\overline{p} \in S$ and $e_i \gg 0$, $\overline{p} \cdot e_i > 0$, hence $\overline{p} \cdot e_i > \overline{p} \cdot (1-\tau)\square e_i$. Notice also that $(1-\tau)\square e_i \ll e_i$. Let x_i such that $x_i \gg (1-\tau)\square e_i$ and $\overline{p} \cdot x_i < \overline{p} \cdot e_i$. For ν large enough, $x_i \geq (1-\tau^\nu)\square e_i^\nu$, $p^\nu \cdot x_i \leq p^\nu \cdot e_i^\nu$, hence, since $(p^\nu, (x_i^\nu))$ is a τ-local equilibrium of $\mathcal{E}(\mathbf{e}^\nu)$, $u_i(x_i) \leq u_i(x_i^\nu)$. At the limit, one gets $u_i(x_i) \leq u_i(\overline{x}_i)$. Now, let ξ_i such that $\overline{p} \cdot \xi_i \leq \overline{p} \cdot e_i$ and $(1-\tau)\square e_i \leq \xi_i$. Since $\overline{p} \cdot e_i > 0$ and $(1-\tau)\square e_i \ll e_i$, there exists a sequence (ξ_i^μ) converging to ξ_i and satisfying $\xi_i^\mu \gg (1-\tau)\square e_i$ and $\overline{p} \cdot \xi_i^\mu < \overline{p} \cdot e_i$ for all μ. From the argument above, $u_i(\xi_i^\mu) \leq u_i(\overline{x}_i)$. So, from the continuity of u_i, at the limit, $u_i(\xi_i) \leq u_i(\overline{x}_i)$, which means that $\overline{x}_i$ is the demand of agent i for the price $\overline{p}$.
$\square$

5 Appendix

Proof of Proposition 3.1. The Taylor formula applied to u_i at e_i gives:

$$\begin{aligned} u_i(e_i + \tau_i\square(x_i - e_i)) - u_i(e_i) = &\nabla u_i(e_i) \cdot \tau_i\square(x_i - e_i) + \\ &\frac{1}{2} D^2 u_i(\xi_i)[\tau_i\square(x_i - e_i), \tau_i\square(x_i - e_i)] \end{aligned}$$

Where $\xi_i \in [e_i, e_i + \tau_i\square x_i]$. Hence:

$$\begin{aligned} u_{ie_i}^{\tau_i}(x_i) = &\nabla u_i(e_i) \cdot (\frac{1}{\|\tau_i\|} \tau_i\square(x_i - e_i)) + \\ &\frac{1}{2\|\tau_i\|} D^2 u_i(\xi_i)[\tau_i\square(x_i - e_i), \tau_i\square(x_i - e_i)]. \end{aligned}$$

Then we have:

$$
\begin{aligned}
|u^{\tau_i}_{ie_i}(x_i) - \hat{u}^{\rho_i}_{ie_i}(x_i)| &= \left\| \frac{1}{2\|\tau_i\|} D^2u_i(\xi_i)[\tau_i \square (x_i - e_i), \tau_i \square (x_i - e_i)] \right\| \\
&\leq \frac{1}{2\|\tau_i\|} \|D^2u_i(\xi_i)\| \|\tau_i \square (x_i - e_i)\|^2 \\
&\leq \frac{1}{2} \|\tau_i\| \|D^2u_i(\xi_i)\| \|x_i - e_i\|^2
\end{aligned}
$$

The last inequality comes from the fact that:

$$
\begin{aligned}
\|\tau_i \square (x_i - e_i)\| = \textstyle\sum_{h=1}^{\ell} |\tau_i^h||x_i^h - e_i^h| &\leq \sqrt{\sum_{h=1}^{\ell} |\tau_i^h|^2} \sqrt{\sum_{h=1}^{\ell} |x_i^h - e_i^h|^2} \\
&\leq \|\tau_i\| \|x_i - e_i\|
\end{aligned}
$$

The subset

$$
\mathbb{K}' = \{(e_i + \tau_i \square (x_i - e_i))_{i=1}^m \mid (\mathbf{e}, \tau, x) \in \mathbf{E} \times [0, 1/2]^\ell \times \mathbf{K}\}
$$

is a compact subset of $\mathbb{R}^\ell_{++}$. So M' and M'' defined by:

$$
M' = \max_i \sup_{\xi \in \mathbf{K}'} \|D^2u_i(\xi_i)\|, \quad M'' = \max_i \sup_{(\mathbf{e},x) \in \mathbf{E} \times \mathbf{K}} \|x_i - e_i\|^2
$$

are finite since u_i is twice continuously differentiable on $\mathbb{R}^\ell_{++}$ for all i. We yield the result by taking $\bar{\theta} \leq \frac{2\varepsilon}{\ell M' M''}$. □

Proof of corollary 3.1

Since u_i is continuous, U_i is continuous on $[0, \frac{1}{2}]^m \times S^m \times \mathbf{E} \times (\mathbb{R}^\ell_+)^m$ if $\theta_i \neq 0$.

We just have to check the continuity of U_i when $\theta_i = 0$. Let $(\bar{\theta}, \bar{\rho}, \bar{\mathbf{e}}, \bar{x})$ such that $\bar{\theta}_i = 0$. Let $\mathbf{K}$ be a compact neighborhood of $\bar{x}$ in $(\mathbb{R}^\ell_+)^m$. Let $\varepsilon > 0$. Since the mapping $(\rho, \mathbf{e}, x) \to \hat{u}^{\rho_i}_{ie_i}(x_i)$ is continuous on $S^m \times \mathbf{E} \times (\mathbb{R}^\ell_+)^m$, there exists an open neighborhood A of $(\bar{\rho}, \bar{\mathbf{e}}, \bar{x})$ such that for all $(\rho, \mathbf{e}, x) \in A$,

$$
|\hat{u}^{\rho_i}_{ie_i}(x_i) - \hat{u}^{\bar{\rho}_i}_{i\bar{e}_i}(\bar{x}_i)| \leq \varepsilon/2.
$$

From Proposition 3.1, there exists $\bar{\theta} > 0$ such that:
for all $(\tau, \mathbf{e}, x) \in ([0, 1/2]^\ell)^m \times \mathbf{e} \times \mathbf{K}$, if $0 < \|\tau_i\| \leq \bar{\theta}$, then

$$
|u^{\tau_i}_{ie_i}(x_i) - \hat{u}^{\rho_i}_{ie_i}(x_i)| \leq \varepsilon/2
$$

with $\rho_i = (1/\|\tau_i\|)\tau_i$.

Let B be the subset of $[0, 1/2]^m \times [A \cap (S^m \times \mathbf{E} \times \mathbf{K})]$ such that $\theta_i \leq \bar{\theta}$. B is a neighborhood of $(\bar{\theta}, \bar{\rho}, \bar{\mathbf{e}}, \bar{x})$ in $[0, \frac{1}{2}]^m \times S^m \times \mathbf{E} \times (\mathbb{R}^\ell_+)^m$ since K is a neighborhood of $\bar{x}$ and A a neighborhood of $(\bar{\rho}, \bar{\mathbf{e}}, \bar{x})$. Let $(\theta, \rho, \mathbf{e}, x) \in B$. If $\theta_i = 0$,

$$|U_i(\theta, \rho, e, x) - U_i(\bar{\theta}, \bar{\rho}, \bar{\mathbf{e}}, \bar{x})| = |\hat{u}^{\rho_i}_{ie_i}(x_i) - \hat{u}^{\bar{\rho}_i}_{i\bar{e}_i}(\bar{x}_i)| \leq \varepsilon/2$$

since $(\rho, \mathbf{e}, x) \in A$.

If $\theta_i > 0$,

$$\begin{aligned}|U_i(\theta, \rho, e, x) - U_i(\bar{\theta}, \bar{\rho}, \bar{\mathbf{e}}, \bar{x})| &\leq |u^{\theta_i\rho_i}_{ie_i}(x_i) - \hat{u}^{\rho_i}_{ie_i}(x_i)| + |\hat{u}^{\rho_i}_{ie_i}(x_i) - \hat{u}^{\bar{\rho}_i}_{i\bar{e}_i}(\bar{x}_i)| \\ &\leq \varepsilon\end{aligned}$$

since $(\rho, \mathbf{e}, x) \in A$ and $0 < \|\theta_i\rho_i\| = \theta_i \leq \bar{\theta}$. □

Proof of Lemma 4.2

For all $p \in \mathbb{R}^\ell_{++}$ and $w \geq 0$, one has:

$$v^{\tau_i}_{ie_i}(p, w) = \max_{\alpha \in S} u^{\tau_i}_{ie_i}\left(\left(\frac{\alpha_h w}{p_h}\right)^\ell_{h=1}\right).$$

and there exists $\alpha \in S$ such that

$$v^{\tau_i}_{ie_i}(p, w) = u^{\tau_i}_{ie_i}\left(\left(\frac{\alpha_h w}{p_h}\right)^\ell_{h=1}\right).$$

Consequently for all $q \in \mathbb{R}^\ell$, $V^{\tau_i}_{ie_i}(q) = \max_{\alpha \in S} V^{\tau_i}_{ie_i\alpha}(q)$, where

$$V^{\tau_i}_{ie_i\alpha}(q) = u^{\tau_i}_{ie_i}\left(\left(\alpha_h \sum_{k=1}^{\ell} e_{ik}\exp(q_k - q_h)\right)^\ell_{h=1}\right).$$

It suffices to show that for all i, there exists $\overline{\theta}_i > 0$ such that for all $\tau_i \in (\Gamma \cap]0; \overline{\theta}_i]^\ell)$, for all $\mathbf{e} \in \mathbf{E}$ and for all $\alpha \in S$, $V^{\tau_i}_{e_i\alpha}$ is strictly convex on the convex hull of Q, co$Q$. One then gets the result by taking the minimum of $\overline{\theta}_i$, $i = 1, \ldots, m$.

For all i, $\hat{E}_i$ denotes the projection of $\mathbf{E}$ on the i-th component of $(\mathbb{R}^\ell)^m$. It is a compact subset included in $\mathbb{R}^\ell_{++}$.

For the sake of simpler notation, we omit the subscript i in the remaining of the proof. Let us compute the Hessian matrix of $V^\tau_{e\alpha}$. For all $\chi \in \mathbb{R}^\ell$, $\chi \cdot D^2V^\tau_{e\alpha}(q)(\chi)$ is equal to[2]

$$\sum_{h=1}^{\ell} \alpha_h u^{\tau\prime}_{eh}(\xi)\left(\sum_{k \neq h} e_k \exp(q_k - q_h)(\chi_k - \chi_h)^2\right) + \sum_{h,k=1}^{\ell} \alpha_h\alpha_k u^{\tau\prime\prime}_{ekh}(\xi)\zeta_h\zeta_k \quad (1)$$

[2]See below for the details of computations.

where

$$\xi = \left(\alpha_h \sum_{k=1}^{\ell} e_k \exp(q_k - q_h) \right)_{h=1}^{\ell}, \; \zeta = \left(\sum_{k \neq h} e_k \exp(q_k - q_h)(\chi_k - \chi_h) \right)_{h=1}^{\ell},$$

$u^{\tau\prime}_{eh}$ is the first order partial derivative of u^{τ}_e with respect to the hth component and $u^{\tau\prime\prime}_{ekh}$ is the first order partial derivative of $u^{\tau\prime}_{eh}$ with respect to the kth component.

From the definition of u^{τ}_e, one remarks that the Hessian matrix of u^{τ}_e at ξ is equal to $\left(\tau^h \tau^k u''_{hk}(e + \tau \square (\xi - e))\right)_{hk}$. Note that for all $\tau \in (]0, \frac{1}{2}])^{\ell}$ and for all $(e, q) \in \hat{E} \times \mathrm{co}Q$, $e + \tau \square (\xi - e)$ remains in a compact set A of $\mathbb{R}^{\ell}_{++}$. Hence, the norm of the Hessian matrix of u is bounded above on A and we denote by $\bar{\delta}$ an upper bound. Consequently, for all $(e, q, \tau) \in \hat{E} \times \mathrm{co}Q \times (]0, \frac{1}{2}])^{\ell}$,

$$\sum_{h,k=1}^{\ell} \alpha_h \alpha_k u^{\tau\prime\prime}_{kh}(\xi) \zeta_h \zeta_k \geq -\bar{\delta} \sum_{h=1}^{\ell} \alpha_h^2 (\tau^h)^2 \zeta_h^2 \geq -\bar{\delta} \sum_{h=1}^{\ell} \alpha_h (\tau^h)^2 \zeta_h^2.$$

The second inequality comes from the fact that $0 \leq \alpha_h \leq 1$ for all h. We remark that

$$\zeta_h = \sum_{k \neq h} \sqrt{e_k \exp(q_k - q_h)} \sqrt{e_k \exp(q_k - q_h)} (\chi_k - \chi_h).$$

Using the Cauchy-Schwartz inequality, one has

$$\zeta_h^2 \leq \left(\sum_{k \neq h} e_k \exp(q_k - q_h) \right) \left(\sum_{k \neq h} e_k \exp(q_k - q_h)(\chi_k - \chi_h)^2 \right).$$

Hence, one obtains the following lower bound for $\chi \cdot D^2 V^{\tau}_{e\alpha}(q)(\chi)$

$$\sum_{h=1}^{\ell} \alpha_h \left[u^{\tau\prime}_{eh}(\xi) - \bar{\delta} (\tau^h)^2 \left(\sum_{k \neq h} e_k \exp(q_k - q_h) \right) \right] \left(\sum_{k \neq h} e_k \exp(q_k - q_h)(\chi_k - \chi_h)^2 \right)$$

Once again, from the definition of u^{τ}_e, since $e + \tau \square (\xi - e)$ remains in A, there exists a lower bound $\underline{\delta} > 0$ such that for all h, for all $x \in A$, $\underline{\delta} \leq u'_h(x)$. Consequently, for all $(e, q, \tau) \in \hat{E} \times \mathrm{co}Q \times (]0, \frac{1}{2}])^{\ell}$, $\tau^h \underline{\delta} \leq u^{\tau\prime}_{eh}(\xi)$.

Let $\delta_h = \max\limits_{(e,q) \in \hat{E} \times \mathrm{co}Q} \sum\limits_{k' \neq h} e_{k'} \exp(q_{k'} - q_h)$ and $\delta = \max\limits_{h=1,\ldots,\ell} \{\delta_h\}$. Then for all $(e, q, \tau) \in \hat{E} \times \mathrm{co}Q \times (]0, \frac{1}{2}])^{\ell}$, we get:

$$\chi \cdot D^2 V^{\tau}_{e\alpha}(q)(\chi) \geq \sum_{h=1}^{\ell} \alpha_h \tau^h \left[\underline{\delta} - \tau^h \bar{\delta} \delta \right] \left(\sum_{k \neq h} e_k \exp(q_k - q_h)(\chi_k - \chi_h)^2 \right)$$

Let $\bar{\theta} < \underline{\delta} / \bar{\delta} \delta$. Hence for $\tau \in (]0; \bar{\theta}])^{\ell}$, for all $(e, q) \in \hat{E} \times \mathrm{co}Q$, $\chi \cdot D^2 V^{\tau}_{e\alpha}(q)(\chi) \geq 0$ for all $(e, q) \in \hat{E} \times \mathrm{co}Q$ and $\chi \cdot D^2 V^{\tau}_{e\alpha}(q)(\chi) > 0$ if $\chi \neq 0$ and $\chi_{\ell} = 0$. Since for all $q \in \mathrm{co}Q$, $q_{\ell} = 0$, one concludes that $V^{\tau}_{e\alpha}$ is strictly convex on $\mathrm{co}Q$ for all $(\tau, e) \in (\Gamma \cap (]0; \bar{\theta}])^{\ell} \times \hat{E}$. $\square$

Computation of formula (1)

Recall that α belongs to the simplex S, q to a compact subset co$Q$ of $\mathbb{R}^\ell_+$ and χ to $\mathbb{R}^\ell$. $V^\tau_{e\alpha}(q)$ and $\varphi_j(q)$ are defined as follow:

$$V^\tau_{e\alpha}(q) = (u^\tau_e \circ \varphi)(q) = u^\tau_e\left(\varphi_1(q), \cdots, \varphi_\ell(q)\right); \quad \varphi_h(q) = \alpha_j \sum_{k=1}^{\ell} e_k \exp(q_k - q_h)$$

Let $\xi = \varphi(q) = \left(\alpha_h \sum_{k=1}^{\ell} e_k \exp(q_k - q_h)\right)_{h=1}^{\ell}$. From the chain rule for the derivatives, one has for all $(\chi, \chi') \in (\mathbb{R}^\ell)^2$,

$$\chi' \cdot D^2 V^\tau_{e\alpha}(q)(\chi) = D^2 u^\tau_e(\xi)\left(D\varphi(q)(\chi); D\varphi(q)(\chi')\right) + Du^\tau_e(\xi)\left(D^2\varphi(q)(\chi,\chi')\right). \tag{2}$$

Note that

$$D\varphi(q)(\chi) = \left(\alpha_h \sum_{k\neq h} e_k \exp(q_k - q_h)(\chi_k - \chi_h)\right)_{h=1}^{\ell} = \alpha \square \zeta \tag{3}$$

and

$$D^2\varphi(q)(\chi,\chi') = \left(\alpha_h \sum_{k\neq h} e_k \exp(q_k - q_h)(\chi_k - \chi_h)(\chi'_k - \chi'_h)\right)_{h=1}^{\ell} \tag{4}$$

Consequently,

$$\begin{aligned} \chi \cdot D^2 V^\tau_{e\alpha}(q)(\chi) &= D^2 u^\tau_e(\xi)\left(\alpha\square\zeta; \alpha\square\zeta\right) + \\ &\quad Du^\tau_e(\xi)\left(\alpha_h \textstyle\sum_{k\neq h} e_k \exp(q_k - q_h)(\chi_k - \chi_h)^2\right)_{h=1}^{\ell} \\ &= \textstyle\sum_{h,k=1}^{\ell} \alpha_h \alpha_k u^{\tau\prime\prime}_{ekh}(\xi)\zeta_h\zeta_k + \\ &\quad \textstyle\sum_{h=1}^{\ell} \alpha_h u^{\tau\prime}_{eh}(\xi)\left(\textstyle\sum_{k\neq h} e_k \exp(q_k - q_h)(\chi_k - \chi_h)^2\right) \end{aligned}$$

References

[1] Balasko, Y., *Fondements de la Théorie de l'équilibre général*, Economica, 1988.

[2] Cornet, B. , *Linear Exchange Economies*, Cahier Eco-Math, CERMSEM, Université Paris 1, 1989.

[3] Bottazzi, J-M. *Accessibility of Pareto optima by Walrasian Exchange processes*. Journal of Mathematical Economics, Elsevier, vol. 23(6), pp. 585-603, 1994.

[4] Champsaur, P. and Cornet, B.: *Walrasian Exchange Processes*. Economic Decision Making : Games, Econometrics, and Optimisation, Elsevier Sciences Publishers B.V.,1990.

[5] Debreu, G., *Economies with a Finite Set of Equilibria*, Econometrica, Vol. 38, pp. 387-392, 1970.

[6] Dierker, E., *Regular Economies* , Handbook of Mathematical Economics, Vol. 1, 1981, PP. 795-830.

[7] Gale, D. *The Theory of Linear Economic Models*. The University of Chicago Press, Chicago 60637, 1960.

[8] Mas-Colell, A., *The theory of general economic equilibrium. A differentiable approach*, Cambridge University Press, 1985.